Submitted to IEEE Transactions on Aerospace and Electronic Systems

Optimum Linear Filtering of an Integrated Signal in White Noise*

Let an analog message, $a(t)$, be passed through an ideal integrator to produce a signal, $b(t)$, which is observed in additive white noise, $n(t)$. Let the observed signal, $r(t)$, defined by:

HC $1.00
MF .50

$$r(t) = b(t) + n(t) = \int_{-\infty}^{t} a(u)\,du + n(t) ,$$

be available over the interval $(-\infty, t)$. The spectra associated with $a(t)$, $b(t)$, and $n(t)$ are rational and will be denoted by $S_a(\omega^2)$, $S_b(\omega^2) = S_a(\omega^2)/\omega^2$, and N_0, respectively. Given the observed signal, a linear, minimum-mean-square-error estimate of the message, $a(t)$, is desired.

Beginning with the solution to the Wiener-Hopf equation, we shall demonstrate:

(i) The optimum linear filter for estimating $a(t)$ without delay is given by:

$$H_{opt}(\omega) = j\omega\sqrt{N_0}\left[1 - \frac{j\omega + f(0)}{\left[S_a(\omega^2) + \omega^2 N_0\right]^+}\right] \tag{1}$$

(ii) The minimum-mean-square error, ϵ^2_{min}, in estimating $a(t)$ without delay is:

$$\epsilon^2_{min} = \frac{N_0}{3} f^3(0) + F(0)$$

N66 29381
(ACCESSION NUMBER) (THRU) 1
(PAGES) 5 (CODE) 07
(NASA CR OR TMX OR AD NUMBER) CR-67853 (CATEGORY)
FACILITY FORM 602

where

$$f(0) = \int_{-\infty}^{\infty} \log\left[1 + \frac{S_a(\omega^2)}{\omega^2 N_0}\right] \frac{d\omega}{2\pi}$$

and

$$F(0) = \int_{-\infty}^{\infty} \omega^2 N_0 \log\left[1 + \frac{S_a(\omega^2)}{\omega^2 N_0}\right] \frac{d\omega}{2\pi} \tag{4}$$

We also note that as a consequence of the results of Yovits and Jackson[1] or Snyder[2] the minimum-mean-square error in estimating $b(t)$ without delay is given by $N_0 f(0)$.

The above model is important because it arises very naturally in the consideration of the optimization and performance of FM demodulators operating above threshold, as pointed out, for example, by Viterbi and Cahn.[3] In this instance, ϵ^2_{min} describes the estimation performance and $N_0 f(0)$ the operational performance of the optimum FM demodulator.

The expression for ϵ^2_{min} is significant because it involves only the known input spectra and does not require a determination of $H_{opt}(\omega)$ for its evaluation. An identical expression has been given by Becker, Chang, and Lawton[4] whose derivation is considerably more involved than that presented here. For the following derivation we closely parallel Snyder.[2]

Derivation of the Expression for $H_{opt}(\omega)$

From the solution to the Wiener-Hopf equation, we have:

$$H_{opt}(\omega) = \frac{1}{\left[S_b(\omega^2) + N_0\right]^+} \left[\frac{j\omega\, S_b(\omega^2)}{\left[S_b(\omega^2) + N_0\right]^-}\right]_+$$

$$= j\omega - \frac{1}{\left[S_b(\omega^2) + N_0\right]^+} \left[\frac{j\omega N_0}{\left[S_b(\omega^2) + N_0\right]^-}\right]_+ \tag{5}$$

where the superscripts "+" and "-" indicate spectral factorization and the subscript "+" indicates taking the realizable part of a partial-fraction expansion. The bracketed expression in (5) with the subscript "+" is a rational function whose numerator is of degree exactly one greater than its denominator. This expression has the form $j\omega k_1 + k_0 +$ [unrealizable terms] when expanded in a partial fraction. We obtain $k_1 = \sqrt{N_0}$ provided $\lim_{\omega\to\infty} S_b(\omega^2) = 0$. Consequently:

$$H_{opt}(\omega) = j\omega - \frac{j\omega\sqrt{N_0} + k_0}{\left[S_b(\omega^2) + N_0\right]^+} \tag{6}$$

We shall prove below that $k_0^2 = N_0 f^2(0)$ and that $\left|\frac{1}{j\omega} H_{opt}(\omega)\right| \leq 1$ at $\omega = 0$ so that k_0 is positive. (1) then follows from (6) by using the fact that $S_b(\omega^2) = S_a(\omega^2)/\omega^2$.

Derivation of the Expression for ϵ^2_{min}

The minimum-mean-square error is given by:

$$\epsilon^2_{min} = \int_{-\infty}^{\infty} \left|1 - \frac{1}{j\omega} H_{opt}(\omega)\right|^2 S_a(\omega^2)\, \frac{d\omega}{2\pi} + \int_{-\infty}^{\infty} \omega^2 N_0 \left|\frac{1}{j\omega} H_{opt}(\omega)\right|^2 \frac{d\omega}{2\pi} \tag{7}$$

From (1) we have:

$$\left|1 - \frac{1}{j\omega} H_{opt}(\omega)\right|^2 = \frac{\omega^2 N_0 + k_0^2}{S_a(\omega^2) + \omega^2 N_0} \tag{8}$$

And from the Appendix we have:

$$\left|\frac{1}{j\omega} H_{opt}(\omega)\right|^2 = \frac{k_0^2 - S_a(\omega^2)}{S_a(\omega^2) + \omega^2 N_0} + 2\left|\frac{1}{j\omega} H_{opt}(\omega)\right| \cos\varphi(\omega) \tag{9}$$

[redacted]

where

$$\frac{1}{j\omega} H_{opt}(\omega) = \left|\frac{1}{j\omega} H_{opt}(\omega)\right| e^{j\varphi(\omega)}$$

Substituting (8) and (9) into (7) we obtain:

$$\epsilon^2_{min} = \int_{-\infty}^{\infty} \left\{ k_0^2 + 2\omega^2 N_0 \left| \frac{1}{j\omega} H_{opt}(\omega) \right| \cos\varphi(\omega) \right\} \frac{d\omega}{2\pi}$$

$$= \int_{-\infty}^{\infty} \left\{ k_0^2 + 2\omega^2 N_0 \frac{1}{j\omega} H_{opt}(\omega) \right\} \frac{d\omega}{2\pi} \qquad (10)$$

We now make the following four observations:

(i) $\left|\frac{1}{j\omega} H_{opt}(\omega)\right| \leq 1$. To prove this assume that $\left|\frac{1}{j\omega} H_{opt}(\omega)\right| > 1$ over some range of frequencies and examine (7). Replacing $\left|\frac{1}{j\omega} H_{opt}(\omega)\right|$ by 1 at these frequencies reduces the mean-square error resulting in a contradiction since ϵ^2_{min} is already minimum.

(ii) $H_{opt}(\omega) \approx k_0^2/2j\omega N_0$ for ω large for otherwise, from (10), ϵ^2_{min} diverges.

(iii) $\int_{-\infty}^{\infty} 2\omega^2 N_0 \frac{1}{n} \left\{ \frac{1}{j\omega} H_{opt}(\omega) \right\}^n \frac{d\omega}{2\pi} = 0$ for $n=2,3,4,\cdots$. A simple application of contour integration shows that the integral is zero for $n=2,3,4,\cdots$ since the integrand is right-half plane analytic and behaves as $1/\omega^{2n-2}$ for ω large.

(iv) $\int_{-\infty}^{\infty} \frac{1}{n} \left\{ \frac{1}{j\omega} H_{opt}(\omega) \right\}^n \frac{d\omega}{2\pi} = 0$ for $n=1,2,3,\cdots$. The proof is identical to that of (iii).

Using these observations and the logarithmic expansion:

$$-\log(1-x) = x + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \cdots \qquad \text{for } |x|<1$$

(10) becomes:

$$\epsilon^2_{min} = \int_{-\infty}^{\infty} \left\{ k_0^2 - 2\omega^2 N_0 \log\left(1 - \frac{1}{j\omega} H_{opt}(\omega)\right) \right\} \frac{d\omega}{2\pi}$$

$$= \int_{-\infty}^{\infty} \left\{ k_0^2 - \omega^2 N_0 \log\left|1 - \frac{1}{j\omega} H_{opt}(\omega)\right|^2 \right\} \frac{d\omega}{2\pi}$$

$$= \int_{-\infty}^{\infty} \left\{ k_0^2 - \omega^2 N_0 \log \frac{\omega^2 N_0 + k_0^2}{S_a(\omega^2) + \omega^2 N_0} \right\} \frac{d\omega}{2\pi} \qquad (11)$$

Also, (iv) leads to:

$$0 = \int_{-\infty}^{\infty} \log\left|1 - \frac{1}{j\omega} H_{opt}(\omega)\right|^2 \frac{d\omega}{2\pi}$$

$$= \int_{-\infty}^{\infty} \log \frac{\omega^2 N_0 + k_0^2}{S_a(\omega^2) + \omega^2 N_0} \frac{d\omega}{2\pi} \tag{12}$$

(11) and (12) can be obtained from the colored-noise results of Yovits and Jackson[1]. In this form, however, ϵ^2_{min} is difficult to evaluate since k_0^2 must first be determined from the integral equation (12). We shall now perform some manipulations which lead to the convenient expression for ϵ^2_{min} given above.

Let

$$f(\lambda) = -\int_{-\infty}^{\infty} \log \frac{\omega^2 N_0 + \lambda}{S_a(\omega^2) + \omega^2 N_0} \frac{d\omega}{2\pi} \tag{13}$$

We seek λ such that $f(\lambda = k_0^2) = 0$. Differentiating (13):

$$\frac{d}{d\lambda} f(\lambda) = -\int_{-\infty}^{\infty} \frac{1}{\omega^2 N_0 + \lambda} \frac{d\omega}{2\pi} = -\frac{1}{2\sqrt{N_0 \lambda}}$$

Integrating and introducing the appropriate boundary condition we then obtain:

$$f(\lambda) = -\sqrt{\frac{\lambda}{N_0}} + f(0) \tag{14}$$

where $f(0)$ is defined by (3). Setting $\lambda = k_0^2$ we then obtain from (14)

$$k_0^2 = N_0 f^2(0) \tag{15}$$

Following the same procedure, let:

$$F(\lambda) = \int_{-\infty}^{\infty} \left\{\lambda - \omega^2 N_0 \log \frac{\omega^2 N_0 + \lambda}{S_a(\omega^2) + \omega^2 N_0}\right\} \frac{d\omega}{2\pi}$$

Then $F(\lambda = k_0^2) = \epsilon^2_{min}$. Differentiating and integrating as before, we obtain:

$$F(\lambda) = \frac{\lambda^{3/2}}{3\sqrt{N_0}} + F(0) \tag{16}$$

where $F(0)$ is defined by (4). Letting $\lambda = k_0^2$ and using (15) we then obtain from (16)

$$\epsilon^2_{min} = \frac{N_0}{3} f^3(0) + F(0) \tag{17}$$

which is the desired result.

Appendix

Let

$$\frac{1}{j\omega} H_{opt}(\omega) = \left| \frac{1}{j\omega} H_{opt} \right| e^{j\varphi(\omega)}$$

Then

$$\left| 1 - \frac{1}{j\omega} H_{opt}(\omega) \right|^2 = 1 + \left| \frac{1}{j\omega} H_{opt}(\omega) \right|^2 - 2 \left| \frac{1}{j\omega} H_{opt}(\omega) \right| \cos\varphi(\omega)$$

Using (8) we easily obtain:

$$\left| \frac{1}{j\omega} H_{opt}(\omega) \right|^2 = \frac{k_0^2 - S_a(\omega^2)}{S_a(\omega^2) + \omega^2 N_0} + 2 \left| \frac{1}{j\omega} H_{opt}(\omega) \right| \cos \varphi(\omega).$$

Donald L. Snyder
Research Laboratory of Electronics
Massachusetts Institute of Technology
Cambridge, Massachusetts

*This work was supported in part by the Joint Services Electronics Program (Contract DA36-039-AMC-3200(E)), the National Science Foundation (Grant GP-2495), and the National Aeronautics and Space Administration (Grants NsG-334 and NsG-496).

References

1. M. Yovits and J. Jackson, "Linear filter optimization with game theory considerations," 1955 IRE National Convention Record, Vol. 3, Part 4, pp. 193-199.

2. D. Snyder, "Some useful expressions for optimum linear filtering in white noise," Proc. IEEE, Vol. 53, No. 6, 629-630; June, 1965.

3. A. Viterbi and C. Cahn, "Optimum coherent phase and frequency demodulation of a class of modulating spectra, IEEE Trans. on Space Electronics and Telemetry, Vol. 10, No. 3, pp. 95-102; 1964.

4. H. Becker, T. Chang, and J. Lawton," Investigation of advanced analog communications techniques," TR. RADC-TR-65-81, Rome Air Development Center, Res. and Dev. Div., Griffiss Air Force Base, N. Y.; March, 1965. Also available as ASTIA Document AD-613703.